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Ancient Greek concept of infinity (ἄπειρον) in the context of the finitistic theory of discretum

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ABSTRACT

The discovery of infinity is an epochal achievement of the ancient Greek philosophy and mathematics. This paper deals with an attempt to uncover the key points of this discovery. Those include the problem of the incommensurability of the diagonal and the side of the square, that is, the problem of the number that could account for this relationships (irrational numbers); Zeno's paradoxes *Achilles and the Tortoise* and *Dichotomy*; Aristotle's considerations concerning the categorematic and syncategorematic infinity with regard to the question of the possibility for a limited, real whole to consist of infinitely many parts. All the analyses will be considered from a specific point of view, i.e. the validity of Archimedes' finitistic axiom in which it is claimed that, regardless of distance, the movement in space from point A to point B can be performed in a finite number of steps, no matter how short these steps might be.

Key words: infinity, finitism, number, continuum, actual, potential, space, time, incommensurability

According to doxographic testimonies, one of the first written records that indicate the beginning of the history of Western European metaphysics is related to the fragment by Anaximander of Miletus. In this fragment there appears a noun that is to cause a lot of 'headache' not only to philosophers and mathematicians, but also to theologians and classical philologists. It is the noun *τὸ ἄπειρον* in the fragment ″ ... *ἀρχή* ... *τῶν ὄντων τὸ ἄπειρον* “. In this fragment *ἄπειρον* (alongside the article) is used as a grammatical substantive, not as an adjective (e.g. compared to the Homeric practice where *ἄπειρον* is used as an attribute of the firmament and earth).[1] The adjectival use would partly neutralize the logically contradictory opposition (limited – unlimited), referring, to a greater degree, to the inability to cross the desert or the sea. The *α-privativum* would refer to the verbal use denoting the lack of transition, the lack of passage. Apparently, *ἄπειρον* is a complicated word which is *prima facie* easy to translate. However, this is not the case at all since it changes the meaning alongside the context even when used by one single author (e.g. Zeno's fragments B1 and B3). It is beyond dispute that *ἄπειρον* does not have an equivalent meaning in Anaximander’s, Zeno’s and Epicurus’s discourse because, for example, in Zeno’s usage the right translation represents the part of the logical proof procedure, not an arbitrary choice. One of the possible translations of the term *ἄπειρον* is *the infinite* since it is (bearing in mind the objective of this paper) the most adequate equivalent in terms of modality. Beside this, it is also legitimate to include translations such as *‘unlimited’*, *‘non-finite’* and sometimes *‘indefinite’* and *‘into infinity’*.[2] It is precisely because we bear in mind previously listed fragments of Zeno (B1 and B3) that it is necessary

[1] In his text *Der Spruch des Anaximander,* the German philosopher Martin Heidegger rejects the Aristotle – Theophrastus’ line of interpretation according to which *ἄπειρον* is a substrate or mediator of the things in existence as well as the Hegelian solution according to which it is regarded as an infinite undeterminedness of being. For Heidegger, *ἄπειρον* is neither infinite nor eternal, nor indeterminate. It is the end point that delivers temporal limits to the presencing (things) that are intent on mere duration (Heidegger 2000, 272).

[2] Ontologically speaking, *ἄπειρον* is neither being nor Being, whereas, formally speaking, if *ἄπειρον* is the *ἀρχή* of (things in) existence, then *ἄπειρον* has no *ἀρχή*. There was an attempt by Anaximander to account for the boundlessness of *ἄπειρον* via temporal phenomena.

for us to distinguish between the two different uses of the term *ἄπειρον*: infinity in terms of number and infinity in terms of magnitude. In the conclusion of B1 it is stated that the beings are such miniscule ones that they lack magnitude and so great that they comprise *ἄπειρα* and are therefore considered infinite in terms of magnitude. Fragment B3, however, deals with infinity in terms of number. The author of the study *Prostor vreme Zenon (Space Time Zeno)* Miloš Arsenijević is of the opinion that the conceptual pair of infinity can give rise to an additional derivation into dynamic and static infinity[3]. His initial hypothesis being that the parts are not created, but limited (discovered) through division, Arsenijević thinks that it is possible to trace the transition from dynamic towards static infinity[4] "… through the procedure in which, starting from the unlimitedness of a certain process, i.e. the process of division that is characterized by succession, one deduces the infinity of the simultaneously existing parts of that whose parts are thus discovered"[5].

Zeno's dialectics is convincing since the initial hypotheses (concerning the existence of the multitude and movement) are refuted through the 'strong' *reductio ad absurdum* in support of Parmenides' claim concerning the existence of One, homogeneous and continuous Being.[6] However, what we are equally concerned with in this paper, beside Zeno's logical proof procedure, is his discovery of infinity via the so called cinematic paradoxes *Achilles and the Tortoise* and *Dichotomy*. The key question that needs to be posed in relation to the race between Achilles and the tortoise (providing the hypothesis concerning the unlimited di-

[3] Arsenijević (1986), 68.

[4] In the proof procedure of the dynamic and static infinity derivation Arsenijević refers back to Fränkel, Vlastos and Abraham who have discovered Zeno's 'arithmetic error' which consists in deducing the infinite magnitude of the things divided from the possibility of an unlimited division.

[5] Arsenijević (1986), 69.

[6] With the exception of the Eleatics, it cannot be claimed that ancient Greek philosophers and mathematicians were overjoyed by the discovery of infinity. The Pythagoreans thought that there existed finitely many natural numbers, whereas Euclid and Aristotle believed in the concept of the potential (syncategorematic) infinity. For Euclid, the straight line was not even considered a mathematical object with no beginning and no end, but an object that can be prolonged 'indefinitely far' (*in indefinitum*).

visibility of space is accepted) is whether Achilles is capable of traversing infinity step by step. If he could traverse an unlimited number of trajectories, that is, reach an unlimited number of points, then the answer to the question would be affirmative. This, however, raises a tricky question of the time in which Achilles' reaching the points takes places. Analogously to *The Arrow Paradox*, in which, according to Aristotle's claim, the arrow never is (temporally speaking) in any of the points of the trajectory throughout its flight, we could say that Achilles reaches the points momentously. Instead of reaching the points, we can analogously talk about the covering of the trajectories of the path. It is possible that the key question is not whether infinity can be covered step by step, but whether continuum (*συνεχές*) can be construed out of discrete indivisible units.[7] Aristotle claims that a continual whole cannot be created out of indivisible elements and that continuum needs to be defined through the continual touching (tangent planes to surfaces of geometrical bodies).[8] There's a sentence by Aristotle that suggests that the infinity cannot be traversed step by step in the finite time span, but it can be traversed in infinity, since the unlimited divisibility of the traversed trajectory implies the infinite divisibility of time.[9] Traversing infinity step by step would be analogous to counting the (infinite) set of natural numbers. The dilemma that has previously been posed is related to the choice between traversing infinity step by step and creating a continuum out of the discrete number of indivisible units. It is actually based on what Aristotle called *Zeno's Axiom* entertaining the proposition that there are no indivisible units of the multitude.[10] An indivisible unit is thought of in terms of a point (*στιγμή*) by Aristotle, that is, in terms of a geometrical concept that lacks magnitude and cannot, for this reason, be a constituent of any multitude since adding one point to another does not result in any magnitude (extension), i.e., that which is added the points does not increase. Of course, the same also holds for other geometrical objects such as the line (*γραμμή*) and planes (*ἐπίπεδον*). However, the formulation of *Zeno's Axiom* does not allow for the points, lines

[7] Arist. *Phys.* 231*a* 25.

[8] Arist. *Metaph.* 1069*a*10.

[9] Arist. *Phys.* 233*a*22-23.

[10] Arist. *Metaph.* 1001*b*7.

and planes to be regarded as having no value at all. They must be thought of as the limits for the entities comprising higher dimensions: the point for the lines, the line for the planes, the planes for the bodies. A pseudo-Aristotelian text *On indivisible lines* (*Περὶ ἀτόμων γραμμῶν*, Lat. *De Lineis Insecabilibus*) contains a principle polemic against that part of Plato's lectures in the Academy that deals with the status of geometrical entities. Ontologically speaking, the problem is related to the way in which a higher genus is contained in a lower one.[11] Aristotle's opinion is that Plato was against regarding the point as the essential geometrical proposition.[12] However, Plato regarded 'the indivisible line' as the principle of a line, the point as the beginning of a line, the line as the beginning of a plane and the plane as the beginning of a body. Ontological negation of the point does not infer a geometrical one or, speaking in contemporary terms, Plato did not accept the theory of juxtaposition in geometry. According to that Platonic interpretation, the line is a kind of Indefinite Dyad (great and small) and is derived form 'long and short'. The plane is derived from 'wide and narrow', whereas the body is derived from 'deep and shallow'.[13] Plato's dualism of the principles (One and Indefinite Dyad) inferred from his unwritten teachings should be understood in the way in which Indefinite Dyad is interpreted as unlimitedly great and unlimitedly small, i.e. as a continuum into infinitely great (> 1) and a continuum into infinitely small (< 1). Plato's continuum of genera, which is also mentioned by Aristotle, relates to the possibility of 'touching' between the genus of the body and that of the plane, between the genus of the plane and that of the line as well as between the genus of the line and that of the point. Namely, one should be very cautious at this point and emphasize that it is one thing to claim that the genera 'touch' and a completely different thing to claim that the higher genera are 'contained' in lower ones. Touching of (the genera of) geometrical entities can also be understood in such a way that the end of one continual thing (e.g., a body) is regarded as the beginning of another

[11] Of course, it is disputable whether this holds for Plato himself or someone belonging to the Academy, for example, Eudoxus or Xenocrates.

[12] Arist. *Metaph.* 992*a*20-24.

[13] Arsenijević (1986), 110.

(e.g., a plane) etc. [14] "… ideality of the line as well as ideality of the plane is precisely the proper reason for their indivisibility. It is only the bodies that are divisible in every respect, whereas indivisibility in one or two dimensions is ascribed to higher genera.[15] In this respect „perceived lines and planes are actually not the real lines and planes and their indivisibility is not spoken of."[16] For the sake of precision, we will emphasize several points in ancient Greek philosophy and mathematics that are directly related to the concept of infinity: the problem of incommensurability, *Achilles and the Tortoise*, *Dichotomy* and *Archimedes' Axiom*.

Incommensurability was discovered by ancient Greek mathematicians while commensurating the diagonal and a side of the square, that is, while looking for a number that could express the relationship between the diagonal and a side of the square. In Plato's dialogue *Theaetetus* (147 d3) a series of irrational numbers $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$ up to $\sqrt{17}$ is explained to the participants by Theodorus of Cyrene. The series of numbers from 3 to 17 cannot be squared in a way that could produce a rational number.[17] The numbers that are the product of multiplication of an equal multiplicand and multiplier are called 'square' or 'equilateral' numbers. The numbers that are called 'rectangular' are those that are produced "either through the multiplication of a bigger number by a smaller one or through the multiplication of a smaller number by a bigger one".[18] Only the numbers that do not contain $\sqrt{2}$ are rectangular. Accordingly, it can be stated that rectangular irrational numbers are only cardinal ones, number 2 excluded. What needs to be emphasized at this point is that (for Theodorus and Theaetetus) the problem of incommensurability is primarily considered a geometrical problem related to the

[14] Arist. *Metaph.* 992*a*13.

[15] Plato favours the expression 'indivisible lines' because the mathematical point is defined as the limit without extension.

[16] Arsenijević (1986), 112.

[17] Arsenijevic thinks that it was a plain fact for the ancient Greeks that there is no "finite number of length units that can give rise to the square of a given magnitude". Also, there is a belief that the discovery of incommensurability was the work of Hippasus of Metapontum (Arsenijević 1986, 45).

[18] Plato (1979), 147d-148a.

incommensurability of a line segment, and the numbers serve the purpose of expressing particular relationships.

The discovery of incommensurability of the diagonal and the side of the square led to the transition from the ostensive version (the one using images) of the Pythagorean proof to the classical indirect arithmetic one (Aristotle). The problem of linear incommensurability emerged alongside Pythagorean attempt to find a geometric mean between two musical intervals (consonants) through which the relation (1,2,3,4) is expressed. If two intervals form the so called *ratio superparticularis* which can be found in the fourth (4:3), fifth (3:2) and octave (2:1), then there is no numerical geometric mean for the numbers that are *ratio superparticularis*. More generally speaking, the Pythagoreans posed a question concerning the conditions under which it is possible to find a geometric mean between two numbers, which is not always possible to do. The question thus formulated influenced the discovery of the linear incommensurability.[19] Euclid's *Elements* (Book X, Theorem 2) provide a method for analyzing the commensurability of the incommensurable magnitudes, but it does not in fact determine the existence of the incommensurable magnitudes in geometry.[20] If *a* and *b* are incommensurable magnitudes, i.e. the magnitudes that do not stand in any finitely determined relation, the question arises as to the possibility of the incommensurability problem being resolved at all. One of the possible solutions for the incommensurability between the magnitudes *a* and *b* to be solved is the introduction of the either infinitely small common measure for *a* and *b* or infinitely big common measure in which a and b would become commensurable, but in infinity. Unfortunately, the finitistic character of the Ancient Greek mathematics was not prone to introducing infinitesimals. "Irrational rectangular numbers could not be commensurated in exponentiation since they cannot be represented as diagonals of the square… The property of irrationality is recursively maintained, and the exponent

[19] Szabo (1978), 176.

[20] This concerns the so called Euclid's algorithm as a variation of the exhaustion method (or the method of approximation, to be more precise) in which the difference between the two surfaces being compared always appears smaller that any fixed difference determined in advance, so that in a subsequent indirect reduction to absurd the surface of a given body is determined.

increases in geometrical progression *ad infinitum*."[21] It is precisely because he supports the theory of indivisible lines (or atomic lengths) that Plato proposes the existence of the smallest possible measurement of commensurable magnitudes. Accordingly, it is possible to propose the existence of indivisible lines as the smallest common measurement of the commensurable elements because the hypothesis of the mathematical commensurability has been accepted. In reality, this is a *petitio principii* proof since the thing to be proved has already been assumed.[22] The concept of infinity is accounted for in a completely different manner starting from Plato's dualism of the principles (One and Indefinite Dyad) derived from his unwritten teachings. Namely, the theory of principles, i.e., predicate logic and the theory of universals, is the one from which particular categories and propositional axioms can be deduced. The nature of infinity (through the subdivision of Indefinite Dyad) is derived by Plato via oppositions smaller-greater, stronger-weaker, because "… there where they are present and where they contribute to the strengthening or weakening, that which participates in them does not stop or end, but continues into infinity of undeterminedness".[23] Characteristically, Indefinite Dyad consists of one facing great (> 1) and one facing small (< 1). Plato even calls it *οὐσία* (essence) "… because neither great-and-small nor greater-and-smaller is limited, but both contain within themselves that which is more-and-less leading into infinity".[24] The point is indivisible, and so is One, whereas the line is regarded, according to the idea of Indefinite Dyad, as some kind of transition. This means that the line is a length envisaged between two points and it has no width. Aristotle defines the line as a continuum of length as well, that is, as a magni-

[21] Bratina (2015), 188.

[22] In her doctoral dissertation *Mathematics in Plato's Philosophy (Matematika u Platonovoj filozofiji)* Bratina Visnja mentions that the (pseudo-Platonic) concept of mathematical incommensurability aims at showing that there is no way for the incommensurability to be rendered into commensurable terms "…except through commensuration with something else, which is also incommensurable, nor is there a way in which an irrational relation can be reduced (not just approximated to) a rational one – for this would mean to properly understand what incommensurability is." (Bratina 2015, 193).

[23] Krämer (1997), 345.

[24] Krämer (1997), 349.

tude which is continuous (uninterrupted) in one direction, width in two directions and volume (depth) in three.[25] The line is a geometrical entity that can be interpreted as a monadic, one-dimensional one,[26] but also as a dyadic one with regard to an infinite extension in both directions.[27] Bearing in mind the problem of infinity, we have pointed to Plato's considerations of the incommensurability that had been discovered before him as well as to the principle Indefinite Dyad. After these, we believe, important notes, in the subsequent discussion we will go back to the historical reconstruction of Zeno's discovery of infinity in the so called cinematic aporias (the paradoxes refuting the possibility of motion) *Achilles and the Tortoise* and *Dichotomy*. Following this, the key discussion will be devoted to the finitism in Ancient Greek mathematics. However, before we delve into that we need to add that the justification for the hypothesis concerning the existence of indivisible line segments is found in the prevention from potentially infinite geometrical division. A line segment is a finite entity. Its infinite division would lead to the transformation of the finite entities into infinite ones, which is considered absurd by the supporters of the indivisible lines theory. Still, this pseudo-argument is based on the logical error *non sequitur* for the fact that infinity cannot be traversed step by step does not imply the existence of indivisible line segments. Since the line segment is an extensive magnitude, it can be divided without the division even being finalized. For this reason geometrical atomists (finitists) do not halt the division at the point as a zero dimensional entity, but at the line segment as a one-dimensional entity. Nicol is of the opinion that a 'confusion' has been made due to the lack of distinction between the ideal and the mathematical. However, there can be no misunderstanding since ideality of the line is the reason for its

[25] Arist. *Metaph.* 1016*b*27-30, 1020*a*13.

[26] In the segment that has just been quoted from *Metaphysica* 1016b7-30 Aristotle talks about "that which is indivisible in one direction" (meaning the line – *γραμμή*). However, he uses the term *ἀδιαίρετον* (indivisible) instead of *συνεχές* (continuum). Bratina has tried to connect the term *συνεχές* with the contemporary mathematical idea of continuum. According to this idea, continuum is the feature of a series that fulfills the condition of compactness. This condition is fulfilled when between "… any two terms … there exist infinitely many 'intermediate terms'" (Bratina 2015, 203).

[27] Euclid, 98.18-20.

indivisibility. Accordingly, there could be no way for Plato to confuse the indivisible line for the ideal one.[28] One of the reasons for the indivisible lines hypostasis is to avoid the impossibility of the transition from the point as a zero dimensional entity towards the straight line as a one-dimensional one. On the other hand, the consequence of the acceptance of indivisible lines is the reduction of geometrical entities to numbers.[29] What is the relationship between indivisible lines and commensurable magnitudes? The supporters of the indivisible lines theory, i.e. (atomic) line segments presume that there exists the smallest common measure for the commensurable magnitudes and that the indivisible line segments comprise the smallest common measure.[30] However, just as there is no possibility for the irrational numbers to be reduced to the rational ones, analogously there is no way for the incommensurability to be commensurated.

As for the paradox *Achilles and the Tortoise* it was not named by Zeno himself, but by Aristotle and Simplicius (who added the tortoise). It states that not even the fastest racer (*τὸ τάχιστον*) can reach the slowest one (*τὸ βραδύτατον*).[31] When discussing Achilles, Aristotle actually introduces dichotomy (*διχοτομία*), i.e., division into two parts, which supports the claim that it is impossible to reach a particular goal since it is impossible to traverse infinity. Bringing the tortoise into the race can be interpreted in several ways. One of them is to, following Arsenijević, claim that the tortoise can "… serve the purpose of the generalization, namely, that it is not possible to even move because it is impossible to arrive anywhere".[32] Another possibility is to 'use' the tortoise in such a way that any position it occupies appears unreachable. Accordingly, in a kind of *staccato* 'running' (throughout the race with Achilles including occasional slowdown) the tortoise will be ascribed a minimal speed and an initial position. However, a legitimate question can be posed whether both time and speed could be brought into play for the purpose of the correct explication of the cinematic aporias. This is not what Aristotle

[28] Nicol (1936), 123.

[29] Arona Marcos (1998), 156.

[30] Euclid X.1.

[31] Arist. *Phys*. 239*b*14-15.

[32] Arsenijević (1986), 82.

does in his presentation of the *Dichotomy* even though he finds fault with Zeno's proof procedure: "… that it is impossible for a thing to pass over or severally to come in contact with infinite things in a finite time."[33] It is impossible for a particular racer, i.e. the slowest one (e.g. the tortoise) to touch all the points in the multitude (of that which is unlimited) the racer encounters on the path in a limited time span. However, it is possible for the tortoise to do this in the unlimited time span and thus traverse infinity.[34] "The passage over the infinite, then, cannot occupy a finite time, and the passage over the finite cannot occupy an infinite time: if the time is infinite the magnitude must be infinite also, and if the magnitude is infinite, so also is the time".[35] In contrast to *Dichotomy* which was presented by Aristotle without 'speed', Achilles' paradox brings speed into play by introducing the slowest racer (the tortoise) and the fastest one (Achilles). The key point in support of the proof for both cinematic aporias is that infinity cannot be traversed step by step (in terms of bringing a particular process to an end, e.g. the process of division). If the time of motion ended, it means this had happened in a limited time span.[36] Aristotle is right when he says that Zeno's proofs against the possibility of movement (*Dichotomy*, *Achilles and the Tortoise*, *Arrow* and *Stadium*) caused a lot of difficulty for those who tried to solve them. We have opted for the analysis of the first two of the cinematic aporias due to their connection to the problem of the discovery of infinity which we have formulated, following Aristotle's interpretation, as reaching (the

[33] Arist. *Phys.* 233*a*23-25.

[34] If the model of time contains the beginning and end, then time is *limited*. If the model contains a particular minimum as the smallest element, then that minimum represents the model of the beginning of time. Analogously, if the model contains the maximal element, then that element indicates the end of the time represented. Providing we imagine that there is the beginning of time, that moment (of the beginning) has no preceding, but only successive point(s). Providing there is the end of time, there is the first moment before the end of time, but no first moment after the end.

[35] Arist. *Phys*. 233*a*32-35.

[36] A certain magnitude was considered continuous by the Ancient Greeks if in its (mathematical) structure it resembled the straight line that was not interrupted in any point, but still in this magnitude a common boundary could be defined where its parts touched – this common boundary is the point.

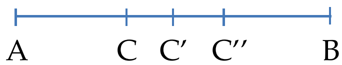
tortoise) and crossing (the trajectories of a path).[37] The *Dichotomy* does not allow for the movement to happen, while Achilles, the fastest racer, is running after the slowest one (the tortoise), still he is running in vain because that which is slowest can never be 'caught' (reached) (*καταληφθήσεται*) by the fastest racer 'chasing' it. The standard (Aristotelian) interpretation of the *Dichotomy* assumes that time and place, being the continuous magnitudes, are unlimitedly divisible without the possibility to be annulled as magnitudes, which means that they can continuously increase (*προέχειν*). The limited time (*πεπερασμένος χρόνος*) that is mentioned by Aristotle in the context of *Achilles* is probably discrete time, i.e., the time limited by ultimate points (*πεπερασμένον τοῖς ἐσχάτοις*).[38] The reason for the parallel analysis is found in the mutual contradiction of the first two aporias, even though they are not internally contradictory. We can list two possible solutions Aristotle offers to Zeno's *Dichotomy*. The first one takes into account the distinction between *ἐνέργεια – δύναμις*, whereas the second proceeds from the thesis that an unlimited number of the trajectories of a path can be traversed in the infinite time. In contemporary versions of the *Dichotomy* a mathematical operation of the infinite series convergence is performed which is also based on Aristotle's formulation of the (proof for) immobility.[39] Accordingly, a racer must first traverse one half of the path before one fourth, eighth and sixteenth of it. This means that the decreasing geometrical progression proceeds from the series: $1/2 + 1/4 + 1/8 + 1/16 \ldots 1/2^n = 1$, whereas the increasing one proceeds from: $1/2 + 3/4 + 7/8 \ldots (2^n - 1)/2^n$. In the decreasing geometrical progression the convergence of the infinite series unlimitedly increases adding one member of the series at a

[37]. "Apart from the formulations in which reaching the goal and transition are mentioned, Aristotle also offers two more formulations. The first one points to the analogy between traversing a path and performing infinitely many successive acts. The second is analogous to counting an infinite set." (Arsenijević 1986, 81).

[38] If there is the time in which there exists a moment for each preceding and/or successive moment, then that time is discrete, regardless the proposition that the time is a continuous magnitude as a part of the category of quantity. No time (in terms of duration) flows in between preceding and successive moments in discrete time.

[39] Arist. *Phys*. 239*b*12.

time being half smaller than the previous one, whereas, on the other hand, the series does not increase unlimitedly since the sum total of the series members is always fewer that one.[40] Comparing *Achilles* and the *Dichotomy*, Aristotle let Achilles run (in the *Dichotomy*) thus proving that he cannot reach the given goal, which is deduced from both proofs "… for in both a division of the space in a certain way leads to the result that the goal is not reached".[41] The difference between the two proofs lies in the facts that the *Dichotomy* deals with the paradox of immobility, whereas *Achilles* deals with unreachability. Independently of the way in which *Dichotomy* is accounted for, the point is that an infinite number of distances cannot be traversed step by step, which can be shown in the 'geometrical' representation of the *Dichotomy* below:



AC+CC′+CC′′+ ... in infinitum = AB

Now we will try to define one aspect we consider relevant for the solution to the problem of Zeno's cinematic aporias, i.e., *FINITISM*. Using *Achilles* as a starting point, a finitist would claim that the path that needs to be crossed by the racer does not contain infinitely many parts, or, in other words, that infinity cannot be traversed step by step. In the study we have already quoted, *Space Time Zeno,* Arsenijević differentiates between finitism and cinematic atomism.[42] He considers finitism an

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[40] "One might propose that the reason for the series convergence lies in the unlimited decrease of the members which ultimately become smaller than any magnitude that could be ascribed to them. That this is not so is shown in the divergence of the so called harmonious series:

1/2 +1/3+1/4+1/5+1/6+1/7+1/8+… It can be seen that the series unlimitedly increases." (Becker 1998, 70).

(1/3+1/4) >1/2 (1/5+1/6+1/7+1/8) >1/2

[41] Arist. *Phys*. 239*b*22.

[42] The advantages of the cinematic atomism, when compared to the geometrical one, is that there is no need for the revision of geometry related the spacial form. The physical atomism of Democritus and Epicurus as well as Plato's geometrical atomism will be set aside for the time being. That which is common for any atomism is an attempt to quantize time and space up to the smallest indivisible magnitude (elementary particles) *τόπος* and *χρόνος*.

advantageous position because it does not postulate any kind of (space) divisibility. However, it denies unlimited divisibility (infinitism) as well as the divisibility to the indefinite extent (indefinitism). According to this interpretation, cinematic atomism is, on the other hand, based on "… the belief in the existence of the 'great disanalogy' between space and time, since it is assumed that even if the facts that the path has been traversed, or time has elapsed do allow for the unlimited division to take place, this does not mean that the movement itself or the time of its origination allow the same thing, too".[43] Apart from this negative finitistic thesis, Arsenijević also focuses on finitism in a narrower sense (positive thesis) which suggests that infinite, convergent series (dichotomies) represent a mathematical construction "…that in itself does not refer to anything real, and still the finitists do not postulate indivisibles in any physical or spacial, or temporal terms."[44] The key question actually is whether continuum extends infinitely, finitely or indefinitely far, or we can, following Zeno, wonder if an infinitely divisible magnitude can be traversed step by step. A finitist would 'allow' Achilles to reach the tortoise for the trajectories of the second half of the (racing) path would, at one point, be treated as those that could no longer be divisible. The arithmetic series through which it is possible to refer to the particular parts of the path and Achilles' running is the following: $[(2^{n-1} – 1) / 2^{n-1}, (2^n – 1) / 2^n]$. The role of the series member 'n' is to specify the finite interval. In finitistic terms, this means indicating the interval that is "in the finite number of times contained in the whole length of the path"[45]. It is not Achilles' task to cross the path, but to reach the tortoise by the end of it. By not postulating indivisibility, the positive thesis of finitism "… still claims that in no case of the limited things, directions, time or movement do we encounter infinity."[46] Also, with the positive finitistic thesis, there is no requirement to talk about the indefinite number of the path trajectories instead of their infinite traversing. The *Achilles* and *Dichotomy* paradoxes are reflected in the impossibility for the limited magnitude which is liable to infinite divisibility to be covered step by step. The problem bigger

[43] Arsenijević (1986), 121.

[44] Arsenijević (1986), 135.

[45] Arsenijević (1986), 135.

[46] Arsenijević (1986), 137.

that Zeno's infinitism is posed with those interpreters who are trying to solve the paradoxes with the claim that a particular trajectory can be covered step by step even though it consists of infinitely many parts. The relationship between the whole and the part can be established in this way if we ask ourselves whether the totality of the path that is to be traversed by Achilles and the tortoise is to be regarded as a *compositum reale*, i.e., as a continuum that consists of the infinite number of actual parts. Provided that we accept the hypothesis that the whole (*compositum*) consists of infinitely many real parts and if we are aiming at, following Aristotle and Kant, the indefinitistic solution to the aporias, then, in the context of Zeno's Achilles, we must accept that it is possible to cover such totality (the length of the path) in the unlimited, and not the limited amount of time, as we have already mentioned.[47] The essential thesis of indefinitism is that the continuum is potentially divisible, but that it is actually divisible indefinitely far (*in indefinitum*). Finitists would refuse to talk about "parts as parts", but would accept the specified parts (one second, one centimeter etc.) In finitistic terms, to cover a trajectory of the path or the whole of the (racing) path means to reach the goal in the finite number of steps, and not to cover infinity step by step. "The number of parts a body has depends on which parts we are talking about. The amount of space a body occupies depends on which temporal interval we are talking about and which referential system we are using. The amount of space can be equivalent to the size of the body, but it can also be larger than the body, even at the same time. Since the identity is relative, there is no contradiction."[48] In fact, Arsenijević thinks that the paradoxes originate due to the absolute determination of the unit. Instead, one should allow for these units to be determined in different ways. It appears that finitism (in the narrow sense of the term) posed no limit to divisibility alongside finitistic and deterministic features of the finitism (as well as atomism). For Arsenijević, this means that a finitist is

[47] Kant is more than explicit in his opting for *compositum ideale* and against *compositum reale*: "Space (as well as time or any magnitude for that matter) should not, in fact, be called *compositum*, but *totum* since its parts are possible only in totality (based on the whole), and it is not the whole that is based on the parts. That would, indeed, be called *compositum ideale*, but not *compositum reale*." (Kant 1976, A438).

[48] Arsenijević (1986), 140.

not able to address the question about the number "of the trajectories counted on the Achilles' path" for one cannot say that this number is indefinite and finite nor can one say that Achilles made "indefinitely finitely many steps".[49] It is necessary for a finitist to simultaneously talk about different number of parts. These parts need to be of a particular kind (any kind) or, to be more precise, there need to exist finitely many parts of a certain kind.

We used Aristotle's *Physics* as a source for the comparative analysis of *Achilles* and *Dichotomy*. However, Aristotle's considerations of infinity are not the direct topic of this paper since it was not Aristotle who discovered infinity, and that he wasn't a finitist, but rather an indefinitist. Namely, Aristotle rejected the theories of discretum (geometrical atomism of the Pythagoreans), the physical atomism of Democritus as well as the theory of the indivisible line that was devised in the Academy. He accepted the theory of continuum that takes into considerations many possible parts of the magnitudes in terms of intervals. There are several reasons why Aristotle supported intervalism. One of them is related to the problem of elementary discontinuity. Another one concerns the structure of time "which is divisible into further divisible parts" since if something is a part of a particular whole, then that part must represent the features of that whole.[50] Finitists do not talk about continuum, but about discretum as limitedly divisible and limitedly divided magnitude. Finitism must be limited and connected to a discrete magnitude that has the limit of its own divisibility for this is exactly what discretum means, i.e. limited divisible magnitude. Finitistic thesis is also shared by the atomists. The difference in the viewpoint consists in the fact that the finitists introduce a relative, whereas atomists introduce an absolute measurement unit. Due to its indeterminedness when it comes to the number of the actual parts finitism approaches the indefinitistic standpoint.

Epicurus thought that that division of space and time leads to absolutely indivisible units of measurement (*τόπος* as a unit of space and *χρόνος* as a unit of time). According to this conception of division, both *τόποι* and *χρόνοι* should be smaller than that which is smallest in per-

[49] Arsenijević (1986), 141.

[50] Arist. *Phys*. 231*b*.

ception. This means that "even though they are extremely small, the final number of *τόποι* is always enough to exhaust any finite space. The same holds for *χρόνοι*, that is, any limited time interval"[51]. Using the example of the least possible 'Egyptian triangle' he indicated the weakness of Epicurus's theory of *χρόνος* "… and regardless of whether the *τόπος* theory is supported or whether space remains unquantized at that."[52] Χ*ρόνοι*, the smallest indivisible units of duration, play their role when it comes to the limited, discrete time which, by definition, has the beginning and the end. Since they are taken to be the smallest indivisible units of time, they need to be understood in terms of (indivisible) moments, and not in terms of (divisible) time intervals. Geometrical atomism of Epicurus should be differentiated from physical atomism supported by Democritus, primarily for those reasons that account for a particular magnitude being called indivisible. Democritus atoms are physically indivisible, whereas, on one hand, Epicurus talks about that which is smallest in perception. On the other hand, in terms of the smallest magnitudes in the third aspect, they can be related to the previous two. That third minimal magnitude would be a particle of an atom, i.e. that which is the smallest particle within an atom. Epicurus's finitism is based on the assumption that even those minimal units, or the particles within an atom, possess a certain magnitude, i.e. a finite magnitude that is limited by great or small. This assumption is used as the proof against infinite division. In that case, these are not just arguments developed in favour of the existence of the minimal indivisibles, but also arguments developed against the possibility of the infinite division process. In his summary of Epicurus's teaching about limited divisibility, Arsenijević adds that it is not only physical atoms that should be labeled as minimal magnitudes, but also geometrical, spacial atoms, i.e. *τόποι*.[53] Does this mean a speculative (theoretical) reading of minimal magnitudes in terms of ends and measure of any other magnitude? While accounting for the movement of atoms, Epicurus introduces the smallest continuous time that has to be understood as a time interval (no matter how minuscule)

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51 Arsenijević (2003), 29.

52 Arsenijević (2003), 29.

53 Arsenijević (1986), 115.

necessary for an atom to leave a space it occupied.[54] "Minimal body parts do not move in minimal time (*χρόνος*); they move neither in the place where they are, nor in the place where they are not. They move by being in two different minimal time intervals (*χρόνοι*) and in two different places (*τόποι*)."[55] Physical atoms have a particular shape, whereas the same thing not only could not be said of the minimal spacial units – *τόποι*, but also one would not be able to see the difference between a geometrical point and *τόπος*. In the light of Epicurus' theory of *χρόνος* and *τόπος*, Arsenijević offers a solution for the *Achilles* and the *Arrow* paradoxes: "Achilles is going to reach the tortoise because he does not have to travers infinity step by step" since "his steps need not be determined by the place the tortoise occupies, but by the time in which they are made …, in one *χρόνος* the arrow is in the spacial interval that has been traversed after the *χρόνος* has elapsed."[56]

Aristotle accounts for the five meanings of the term infinity: infinity of the magnitude in the process of division; infinity of the number in successive addition of one to one; infinity (the sea of the ocean), i.e. that which is difficult to cross (infinity according to the hyperbole); infinity exemplified by a circle, i.e. of that which can be traversed, but that has no limit (the beginning and the end); the infinity of the continuum in terms of that in which infinite magnitudes are demonstrated or manifested exemplified by motion and time.[57] Aristotle accounts for two senses of the term *continuum*: it signifies an immediate concatenation of two things (the end of the former is the beginning of the latter). It can also refer to that which is unified and homogeneous (concatenation of the potential parts of a certain thing). A continuous magnitude cannot consist of indivisible parts since "… any magnitude is further divisible into magnitudes", and the magnitude is uninterrupted.[58] If we observe the two racers (Achilles and the tortoise) performing constant motion, but at a different speed, then a faster racer (Achilles) needs to cover a

[54] Speaking in Aristotelian terms, time is continuous because the present time represents a common boundary for the past and present.

[55] Arsenijević (1986), 116.

[56] Arsenijević (1986), 122.

[57] Arist. *Phys*. 204*a*5-204*b*.

[58] Arist. *Phys*. 232*a*25.

longer path in comparison to a slower one (the tortoise) in the same amount of time. The amount of time determines the racers being faster or slower and it is also obvious that the faster racer will traverse the same distance in a shorter time interval. Besides, the faster racer will not only cover a longer distance than a slower one, but will also do that in a shorter time span. The path that is covered by the 'racers' is always a continuous one and this accounts for a continuous motion, hence Aristotle switches from continuity in terms of the path to continuity in terms of motion. Time as an attribute of the continuous motion is itself continuous because the body trajectory (of the racers Achilles and the tortoise) is an uninterrupted line. “But what is moved is moved from something to something, and all magnitude is continuous. Therefore the movement goes with the magnitude. Because the magnitude is continuous, the movement too is continuous, and if the movement, then the time; for the time that has passed is always thought to be as great as the movement.”[59] Movement is continuous if the path, or the trajectory traversed by the racer in physical space is continuous as well, that is, if the trajectory is an uninterrupted line. The proposition which holds that if the movement is, but rather the assumption in favour of movement over other types of change, such as intensive or qualitative changes, increase and decrease, origination and disappearance, etc. The continuity of movement is, in fact, a metaphysical principle and postulate, not an abstraction of than which is given in experience. The questions Aristotle focused on were concerned with the infinity of the continuum, the potential (syncategorematic) and actual (categorematic) as well as with the infinite magnitudes such as time, space, motion etc. which are also continuous, and these continuous magnitudes are further divisible into parts – *in indefinitum*, i.e. indefinitely far.

The fully formed and mature stage of development of the Ancient Greek mathematics (Eudoxus, Euclid and Archimedes) has a finitistic (anti-infinitistic) character. However, it is indefinitistic according to its presuppositions concerning the nature of the continuous magnitudes. The discovery of incommensurability led to the problem connected with the forms of this theorem when generally applied.[60] Incommensurability

[59] Arist. *Phys*. 219*a*10-15.

[60] Following the discovery that “a square root above the hypotenuse equals the sum of the squares above the both of the legs”, the hypotenuse above the

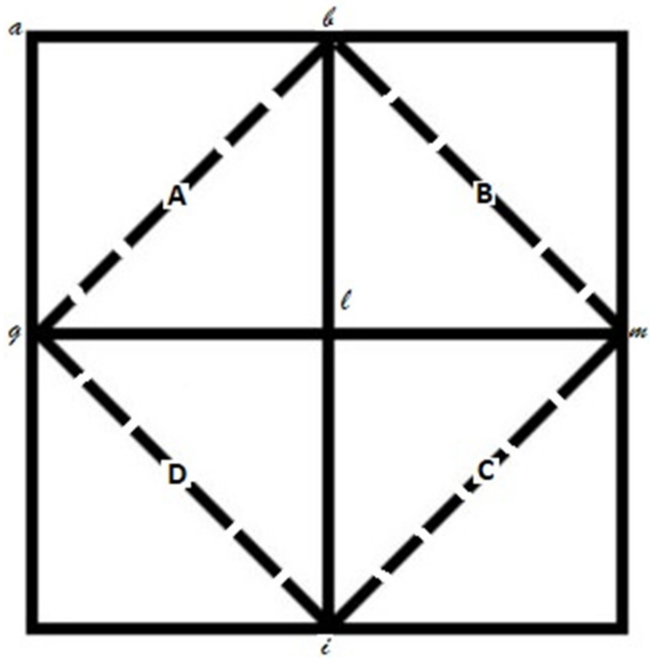
of the diagonal and the side of the square was discovered due to an impossibility to derive the square root of the numbers 2, 3, 5, 7 etc. These numbers represent the measure of the square surface whose length measure multiplied by itself results in the numbers 2, 3, 5, 7. The measure of the side length out of which one can construe a square whose surface consists of a series of four unit lengths comprises two geometrical unit lengths. The square root of the side whose length measurement comprises two length units cannot be calculated above the hypotenuse of the equilateral right-angled triangle. One cannot find the number a whose multiplication by $\sqrt{2}$ results in number 2. Each quotient of the rational number 2 and an irrational number gives rise to an irrational number. The quotient can be calculated by multiplying the diagonal of the square by itself or through the addition of the square planes above the triangle's legs.

In his estimation of the achievements in mathematics dating back to the time of the Pythagoreans Knorr suggested the reconstruction of the 'authentic' incommensurability proof. The main source of his reconstruction of the original Pythagorean proof of incommensurability is Plato's *Meno*.[61] Beginning with the mathematical methodological practice of the Pythagorean time, Knorr thinks that the proof must have had a diagram, i.e. that the proof must have been ostensive. The diagram from *Meno* is combined by Knorr with the indirect arithmetic proof from Aristotle's *First Analytics*. The diagram is represented in the following way:[62]

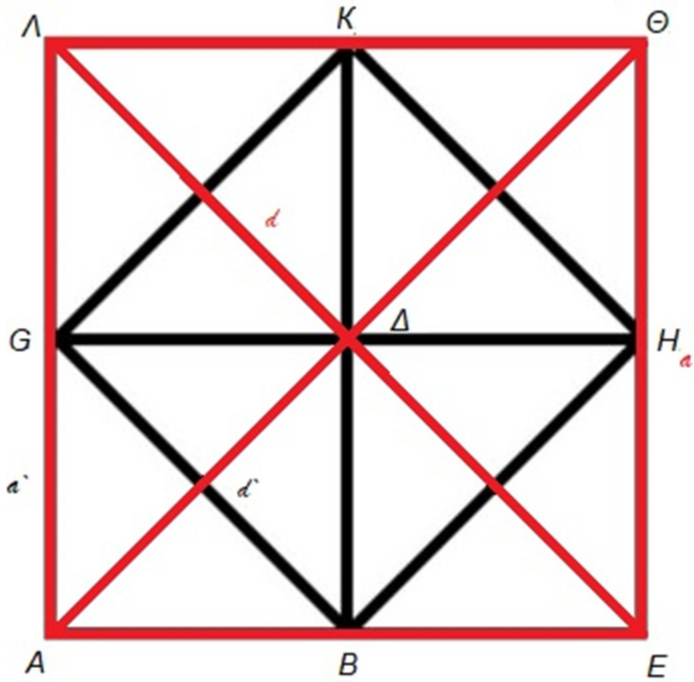
legs of an equal length might have been taken into consideration within the Pythagorean brotherhood. In other words, it could be the case of the hypotenuse contained in the square diagonal, not the rectangle one as was the case of the 'Egyptian triangle' we have mentioned before. Namely, the 'Egyptian triangle' has a commensurable square above the hypotenuse. The problem is that it is not easy to find a unit length by which the sizes of the hypotenuse and the legs of the equilateral right angled triangle can be commensurated. However, regardless of how much the unit length decreases the smallest common measure is not easy to find.

[61] Pl. *Men*. 84d1-85b5.

[62] Knorr (1976), 23-25.



Plato's Diagram



Knorr's Diagram

Based on this representation Knorr poses the question: how many times is the diagonal GB contained in the side AE, GH of the square AEθλ. [63] Provided that the side GB and the diagonal GH were commensurable each would represent a particular multiple of their smallest common measure (*Elem*. VII.14 and VII.15). Knorr suggests that the relation between the diagonal and the side should be treated as that holding between mutual prime numbers, i.e. as those which have no common divisor but number one. Similarly, the squares AEθλ and GBHK can be expressed in terms of square numbers that are the result of the multiplication including two equal factors. The diagram shows that the square AEθλ is a duplicated square GBHK. There follows that the square number of the bigger square is an even number which means that its side GH must be an even side. GB and GH are even numbers, i.e. they are not mutual prime numbers. From this Knorr draws the conclusion that the initial hypothesis is not true, which means that, according to reduction ad absurdum, the line segments GH and GB must ne incommensurable.[64] One also needs to take into account the possibility that Knorr's reconstruction of the original proof for incommensurability is in fact a geometrical interpretation of the early Pythagorean proof that might have been both arithmetic and ostensive one. [65]

Eudoxus of Cnidus (Academy member), whose definition we are about to quote was included into Book V of Euclid's *Elements*, holds the opinion that a particular circle and a particular square are in an equal relationship to some other circle or square although both the former and the latter relationship is that of incommensurability. "Magnitudes are said to be *in the same ratio,* the first to the second and the third to the fourth, when, if any equimultiples whatever are taken out of the first and third, and any equimultiples whatever taken of the second and

[63] Knorr (1975), 26-28.

[64] Ibid. 27.

[65] There can be no doubt that the rejection of the ostensive (empirical) proof in the ancient Greek mathematics was linked to the discovery of the linear incommensurability that needs to be looked for in the Pythagorean theory of the harmony. Mathematical derivation of the linear incommensurability was laid out in definition X.1 in Euclid's elements. The criteria for the commensurability and incommensurability are given in theorems X.5, X.6, X.7, X.8.

fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order."[66] In other words, if in a proportion *a:b = c:d* either both numerators or both denominators are multiplied by the same number, equality will be preserved for it is enough to state that *a* and *b* are in the same relationship as *c* and *d* even if *a* and *b* and/or *c* and *d* are mutually incommensurable magnitudes and/or irrational numbers. The problem of determining the geometric mean of two numbers led to the discovery of the linear incommensurability that was exposed in Book X.1 of Euclid's *Elements*: „Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure." In book *Komentari uz Euklidove Elemente (Commentaries on Euclid's "Elements"*), the translator and commentator Bilimović notes that Euclid's concepts of commensurability and incommensurability are broader than contemporary terms related to the (in)commensurability according to length, in one dimension (p. 141). Theorems 5, 7, 8 of the Book X of the *Elements* list the criteria of commensurability and incommensurability of magnitudes.[67] If there is a possibility to arithmetically express the ratio between two magnitudes, then these magnitudes are commensurable and this depends on having a common measure between them (geometric mean).[68] Magnitudes that are exponentially incommensurable are also incommensurable in terms of length. We cannot claim the certain knowledge of what the proof for incommensurability of the diagonal and the side of the square looked like. It was probably given in ostensive terms (as an image given in per-

[66] Euclid V.5.9.

[67] X.5: „Commensurable magnitudes have to one another the ratio which a number has to a number."(p.14).

X.7: „Incommensurable magnitudes do not have to one another the ratio which a number has to a number." (p.15).

X.8: „If two magnitudes do not have to one another the ratio which a number has to a number, then the magnitudes are incommensurable." (p. 15).

[68] "It is important to note that any incommensurability proof that is based on recursive subtraction represents an indirect proof. Speaking in ostensive terms, it is not clear whether recursive subtraction in case of incommensurable planes ever finishes." (Bratina 2015, 16).

ception) and not as an indirect proof (given in thought, i.e. reflection). In Euclid's *Elements* one cannot find such proof, neither in a geometric nor in an arithmetic version of the ostensive proof.[69] We think that the incommensurability of the diagonal of the square and its side is determined arithmetically, but still through the use of numerical line as an auxiliary geometrical means. The measure of the length (geometrical number) of the side from which the square is construed whose plane measure comprises a series of four geometrical units equals two geometrical units (two line segment units). However, the square of the side whose measure of length comprises two geometrical units cannot be derived above the hypotenuse of the isosceles right triangle. The question now is which number it is that in the function $2a^2$ has the value 4.

$2a^2 = 4$

$a = \sqrt{2}$

Number a will be an irrational number as the quotient of number 2 being the dividend and the square root of number 2 being the divisor. It is one thing to look for the side of a square based on the surface of that square, however, it is another thing to look for the length of the leg of the isosceles right triangle whose hypotenuse would be concordant with the side of that square whose surface is known. The square of the surface of any 4 geometrical units is not possible in terms of the square above the diagonal of a preceding, smaller square. However, it is possible independently, i.e. standing by itself. The squares of the surface comprising 6, 8, 10 etc. geometrical units are not possible in any way for their square is not a rational number. Eleatics considered the possibility of calculating the surface of the circle through the inscription of small triangles and subsequent addition of the surfaces of the inscribed triangles. They claimed that the smaller the triangles inscribed, i.e. the bigger their number within the circle, the more precise measure of the surface will be arrived at. The sum of the infinitely many infinitely small triangles inscribed in the circle will result in the precise surface of the circle. This

[69] The phenomenon of the incommensurable magnitudes represents an argument against the theory of indivisible lines. In the pseudo-Aristotelian text *On indivisible lines*, the presupposition concerning the existence of indivisible lines should have served against the possibility of traversing infinity step by step in a limited time span. The function of indivisible lines was to prevent the infinite geometric division in syncategorematic terms.

was the stepping stone of the infinitesimal mathematics that originated much later as the beginning of mathematical analysis. Without the concept of infinity there would be no infinitesimal calculus. The German mathematician Georg Cantor used the so called 'diagonal argument' to show that there are two different kinds of infinity, i.e. the infinity of natural numbers (discretum, countable infinity) and the infinity of real numbers (continuum). Cantor's theorem proves that there is infinity bigger than continuum, that is, that for each infinity there is an infinity bigger than that one. Through the uninterrupted addition of one to one the countable infinity of the natural and rational numbers is reached. However, continuum is realized only through the irrational numbers. The supporters of the continuum theory think that there is no atomic line segment, i.e. that through the parallel placement (juxtaposition) of two or more points the line cannot construed.

Putting aside the problem of the authenticity of the so called Archimedes' Axiom stating that "... the two magnitudes can be compared provided that there is a smaller finite product that can reach and surpass a greater one", it is beyond doubt that he is the icon of finitism. If for no other reason than the following one, his finitism is based on the methodological supposition that "... the number of inscribed and circumscribed figures must always be finite and therefore there must always exist the difference between the fields derived from inscribing and circumscribing and those derived from an initial figure where the remaining difference must be added so that the initial field could be exhausted…".[70] The axiom which has just been listed, although it might not belong to Archimedes, was accepted by him in a different formulation as well. This means that within a spacial unity one can reach point B starting from point A in a finite number of steps, regardless of distance and no matter how short the steps might be. Archimedes rejected Epicurus's geometrical atomism, but he accepted, according to Aristotle, Zeno's axiom: "Further, if absolute Unity is indivisible, by Zeno's axiom it will be nothing".[71] In other words, line cannot be constituted out of points, that is, the entities of a higher dimension do not consist of the entities belonging to lower dimensions. Lines and planes do not increase that to which they are

[70] Arsenijević (1986), 203.

[71] Arist. *Metaph.* 1001*b*7.

added – if they are concordant, and not concatenated. The point being the zero-dimensional entity, it cannot produce a one-dimensional one through concatenation, adding one point to another.[72] The theory of limited entities aims at demonstrating that the entities of a lower dimension are the limits of the next dimension of a higher order: the plane is the limit of a body, the line is the limit for the surface, etc. Aristotle thought that the lower dimensions conceptually precede the higher ones in the sense that the knowledge of the higher dimensions is based on the lower ones, but that there must be a differentiation between priority in terms of the concept and *οὐσία*.[73] For Aristotle, geometrical objects are abstractions of the physical ones and they do not exist in reality, that is, independently of the physical objects. Based on these assumptions, Arsenijević asks a justified question about how Archimedes dared treat planes as if they consisted out of many things lines.[74] The answer lies in the fact that if Zeno's Axiom is denied and Epicurus's atomism is rejected, the only option left is the introduction of infinitesimals as a part of the proof procedure. Unfortunately, there is a lack of authentic testimony that the Ancient Greek mathematicians used the infinitesimal method in the calculation of a surface or volume. If we take that (geometrical) infinitesimals are magnitudes that are infinitely many times smaller than the limited ones accessible to us, then the previous conclusion is even more convincing. By definition, infinitesimals are simply indivisible. Zeno's Axiom infers that a line cannot be construed out of Euclidian points since, according to Euclid: „A point is that which has no part.“[75] Similarly, Zeno claims that that which has no magnitude (no dimension) cannot constitute any magnitude (a point or line) as well as that a certain lim-

[72] Ancient Greek mathematicians used the methods of approximation in calculating the value of a particular irrational number. Using a progressive increase of the number of polygon sides, Archimedes could approximate the value of the number π. This, of course, does not mean that an irrational number can be reduced to a rational one.

[73] Arist. *Metaph*. 1077*b*1.

[74] Arsenijević (1986), 205.

[75] Euclid I.1.

ited object cannot consist of infinitely many parts[76]. The point being the limit, it must be without parts just as the line is indivisible in terms of breadth and the plane in terms of depth. Introduction of infinitesimals as infinitely small magnitudes would lead to the negation of Archimedes' Axiom "… since any distance, no matter how small or finite, … can be exhausted only in an infinite number of infinitesimals"[77]. The possibility to immediately postulate the connection between small and great sets of numbers as well as points and line segments rendered Archimedes' Axiom invalid due to the existence of the points that cannot be reached in a finite number of steps no matter how big these steps might be.

At the end of this paper we can summarize: Zenon's axiom, which consists in the claim that there is no indivisible unit of the mass, in its mathematical interpretation, is a classic example of infinitism. According to this point of view, the points are null and void in that no line can be made from them. Any size, however small, must be divisible *in infinitum.* At the same time, this position is a direct negation of the geometric

[76] Bilimović, the translator and commentator of Euclid's *Elements*, thinks that Definition 1 in Book 1 can also be translated as "A point is that which has no extension". Indeed, he admits that in the main part of the quoted text Branislav Petronijević's translation was used. This intervention infers that the point is denied both length and breadth by Euclid. Since this is the correct interpretation, an alternative translation has been proposed: "A point is that which has no dimensions."

[77] Arsenijević (2003), 30. The Axiom, which is ascribed to Archimedes or could have belonged to him as well for that matter, is against any introduction of infinitely small magnitudes. However, Archimedes himself used infinitesimals, allegedly for heuristic purposes "… so as to subsequently perform an immediate deduction of the proof using his well known procedure of double reduction to absurd which is in absolute compliance with the rigorous rules of the ancient Greek mathematics in which infinitesimals were not acknowledged." (Arsenijević 2003, 31) Beside the rejection of Archimedes' Axiom, Arsenijević also draws attention to the rejection of Cantor-Dedekind Axiom ("… which supports the claim concerning the straightforward mapping between the numbers of the standard field and the points of the line segments…"). This Axiom is no longer viable because the straightforward mutual mapping (biunivocal correspondence) infers that the line contains more points "… and this includes all those points corresponding to the numbers that stand in a closer relationship to standard numbers than other standard numbers." (Arsenijević 2003, 33).

(Pythagorean, Epicurean) and physical (Leucippus, Democritus) atomism, as well as finitism. If we allow infinite division in mathematical terms, then physical partitioning should not be forbidden. Indefinitists (Aristotle and Kant) acknowledge potential infinity, but deny the current partitioning of the continuum, or infinity in a categorematic sense. The division for them goes indefinitely far (*in indefinitum*).

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[i] T.N.: All quotations from Aristotle's *Physics* are rendered according to *Complete Works (Aristotle) Volume I*, ed. by Jonathan Barnes, Princeton University Press, Princeton, N.J. 1991. The translation of the quotations related to Euclid's *Elements* are taken from https://mathcs.clarcku.edu, Euclid's *Elements* Book V and Book X. Aristotle's *Metaphysics* Book III has been quoted according to *Aristotle in 23 Volumes,* Vols. 17, 18, translated by Hugh Tredennick, Harvard University Press – William Heinemann Ltd. Cambridge, MA – London, 1933, 1989 (Online edition www.perseus.tufts.edu).